$\therefore \exists\ t_k, t_k' \in [x_{k-1}, x_k]$ such that

$f(t_k) - f(t_k') > M_k(f) - m_k(f) - h$

$\Rightarrow f(t_k) - f(t_k') + h > M_k(f) - m_k(f) \longrightarrow (2)$

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For $P \supseteq P_\epsilon$,

Consider

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\,\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\,\Delta\alpha_k$$

$$= \sum_{k=1}^{n} [M_k(f) - m_k(f)]\,\Delta\alpha_k$$

$$< \sum_{k=1}^{n} (f(t_k) - f(t_k') + h)\,\Delta\alpha_k \quad \text{By (2)}$$

$$= \sum_{k=1}^{n} [f(t_k) - f(t_k')]\,\Delta\alpha_k + \sum_{k=1}^{n} h\,\Delta\alpha_k$$

$$= \sum_{k=1}^{n} [f(t_k) - f(t_k')]\,\Delta\alpha_k + h\sum_{k=1}^{n} \Delta\alpha_k$$

$$\leq \left|\sum_{k=1}^{n} [f(t_k) - f(t_k')]\,\Delta\alpha_k\right| + h\,|\alpha(b) - \alpha(a)|$$

$$< \frac{2\epsilon}{3} + \frac{\epsilon}{3\,[\cancel{\alpha(b) - \alpha(a)}]}\,[\cancel{\alpha(b) - \alpha(a)}]$$

$$= \frac{2\epsilon}{3} + \frac{\epsilon}{3}$$

$$= \epsilon$$

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\therefore$ from case (a) and case (b)

'f' satisfies Riemann condition w.r.t α on $[a,b]$

$\therefore$ Statement (ii) is true.

To prove : (ii) $\Rightarrow$ (iii)

Assume that (ii) holds

To prove that : statement (iii) is true

(ie) Assume that f satisfies Riemann condition w.r.t α on $[a, b]$

(ie) $0 \le U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \longrightarrow$ ③

To prove that : $\underline{I}(f, \alpha) = \bar{I}(f, \alpha)$

By theorem : 12 we have assume that

$$\bar{I}(f, \alpha) \ge \underline{I}(f, \alpha) \longrightarrow (I)$$

To prove : $\underline{I}(f, \alpha) \le \bar{I}(f, \alpha)$, it is enough to prove that

$\bar{I}(f, \alpha) \le \underline{I}(f, \alpha)$

$$\begin{aligned}
\bar{I}(f, \alpha) &\le U(P, f, \alpha) \\
&< L(P, f, \alpha) + \epsilon \quad \text{by ③} \\
&\le \text{l.u.b}\{L(P, f, \alpha) : P \in \mathcal{P}[a, b]\} + \epsilon \\
&= \underline{I}(f, \alpha) + \epsilon
\end{aligned}$$

$\bar{I}(f, \alpha) \le \underline{I}(f, \alpha) + \epsilon$

Since $\epsilon > 0$ is arbitrary

$$\bar{I}(f, \alpha) \le \underline{I}(f, \alpha) \longrightarrow \text{II}$$

$\therefore$ From (I) & II

$$\bar{I}(f, \alpha) = \underline{I}(f, \alpha)$$

(ie) statement (iii) is true.

To prove : (iii) $\Rightarrow$ (i)

Assume that statement (iii) holds.

To prove that: Statement (i) is true

(ie) Assume that $\bar{I}(f, \alpha) = \underline{I}(f, \alpha)$

To prove that : $f \in R(\alpha)$ on $[a,b]$

Let $\underline{I}(f,\alpha) = \overline{I}(f,\alpha) = A$

Now $\overline{I}(f,\alpha) = g.l.b \{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

$\therefore$ For any given $\epsilon > 0$.

$\overline{I}(f,\alpha) + \epsilon$ is not a lower bound of $\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

$\therefore$ There exists a partition P_ϵ' of $\mathcal{P}[a,b]$

Such that,

$$U(P_\epsilon', f, \alpha) < \overline{I}(f,\alpha) + \epsilon$$

For every $P \supseteq P_\epsilon'$

$$U(P, f, \alpha) \leq U(P_\epsilon', f, \alpha) < \overline{I}(f,\alpha) + \epsilon$$

$\Rightarrow$ For $\forall\ P \supseteq P_\epsilon'$

$$U(P, f, \alpha) \leq \overline{I}(f,\alpha) + \epsilon \longrightarrow ①$$

Consider,

$$\underline{I}(f,\alpha) = l.u.b \{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$$

$\therefore$ For any $\epsilon > 0$

$\underline{I}(f,\alpha) - \epsilon$ is not an upper bound of the set $\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$.

$\therefore \exists$ a partition P_ϵ'' of $[a,b]$ such that

$$L(P_\epsilon'', f, \alpha) > \underline{I}(f,\alpha) - \epsilon$$

For every $P \supseteq P_\epsilon''$,

$$L(P,f,\alpha) \geq L(P_\epsilon'', f, \alpha) > \underline{I}(f,\alpha) - \epsilon$$

$\therefore \forall\ P \supseteq P_\epsilon''$,

$$L(P,f,\alpha) > \underline{I}(f,\alpha) - \epsilon$$

i.e) $\underline{I}(f,\alpha) - \epsilon < L(P,f,\alpha) \longrightarrow ②$

Let $P_\epsilon = P_\epsilon' \cup P_\epsilon''$, then $P_\epsilon \supseteq P_\epsilon'$ and $P_\epsilon \supseteq P_\epsilon''$

then $P \supseteq P_\epsilon \Rightarrow P \supseteq P_\epsilon'$ and $P \supseteq P_\epsilon''$

for such P inequalities (1) and (2) hold good

$\therefore \forall P \supseteq P_\epsilon$, from (1) and (2)

$\underline{I}(f,\alpha) - \epsilon < L(P,f,\alpha) \le S(P,f,\alpha) \le U(P,f,\alpha) < \bar{I}(f,\alpha) + \epsilon$

[$\because L(P,f,\alpha) \le S(P,f,\alpha) \le U(P,f,\alpha)$]

$\Rightarrow \underline{I}(f,\alpha) - \epsilon < S(P,f,\alpha) < \bar{I}(f,\alpha) + \epsilon$

$\Rightarrow A - \epsilon < S(P,f,\alpha) < A + \epsilon$

$\Rightarrow -\epsilon < S(P,f,\alpha) - A < \epsilon$

$\Rightarrow |S(P,f,\alpha) - A| < \epsilon$, $\forall P \supseteq P_\epsilon$ of $[a,b]$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$ and $A = \int_a^b f\,d\alpha$

i.e.) Statement (i) is true.

COMPARISON THEOREMS

Theorem : 14

Assume that α increasing on $[a,b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$ and if $f(x) \le g(x)$, for all x in $[a,b]$. Then we have,

$$\int_a^b f(x)\, d\alpha(x) \le \int_a^b g(x)\, d\alpha(x)$$

Proof: Given that : (i) α is increasing on $[a,b]$

(ii) $f \in R(\alpha)$ on $[a,b]$ and $g \in R(\alpha)$ on $[a,b]$

To prove that : If $f(x) \le g(x)$ on $[a,b]$, then

$$\int_a^b f\, d\alpha \le \int_a^b g\, d\alpha$$

Consider, $f(x) \le g(x)$, $\forall x \in [a,b]$

then $f(t_k) \le g(t_k)$, $\forall t_k \in [x_{k-1}, x_k]$

Since α is increasing on $[a,b]$; $\Delta\alpha_k \ge 0$

$f(t_k)\,\Delta\alpha_k \le g(t_k)\,\Delta\alpha_k$ $\quad x \in [a,b]$

$t_k \in [x_{k-1}, x_k]$

$$\Rightarrow \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k \le \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k$$

(43)

$$\Rightarrow S(P,f,\alpha) \le S(P,g,\alpha) \longrightarrow (1)$$

Since $f \in R(\alpha)$ on $[a,b]$.

For any given $\epsilon > 0$, we can find a partition P_ϵ' of $[a,b]$ such that $\forall\ P \supseteq P_\epsilon'$ we have,

$$\left| S(P,f,\alpha) - \int_a^b f\,d\alpha \right| < \epsilon/2$$

$$\Rightarrow -\frac{\epsilon}{2} < S(P,f,\alpha) - \int_a^b f\,d\alpha < \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b f\,d\alpha - \frac{\epsilon}{2} < S(P,f,\alpha) < \int_a^b f\,d\alpha + \frac{\epsilon}{2} \longrightarrow (2)$$

Since $g \in R(\alpha)$ on $[a,b]$

For any given $\epsilon > 0$, $\exists$ a partition P_ϵ'' of $[a,b]$ such that $\forall\ P \supseteq P_\epsilon''$ we have,

$$\left| S(P,g,\alpha) - \int_a^b g\,d\alpha \right| < \frac{\epsilon}{2}$$

$$\Rightarrow -\frac{\epsilon}{2} < S(P,g,\alpha) - \int_a^b g\,d\alpha < \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b g\,d\alpha - \frac{\epsilon}{2} < S(P,g,\alpha) < \int_a^b g\,d\alpha + \frac{\epsilon}{2} \longrightarrow (3)$$

let $P_\epsilon = P_\epsilon' \cup P_\epsilon''$

Then $P \supseteq P_\epsilon \Rightarrow P \supseteq P_\epsilon'$ and $P \supseteq P_\epsilon''$

$\therefore$ For such P, inequalities (2) and (3) hold good

Hence, $\forall\ P \supseteq P_\epsilon$, combining (1), (2), (3) we have,

$$\int_a^b f\,d\alpha - \frac{\epsilon}{2} < S(P,f,\alpha) \le S(P,g,\alpha) < \int_a^b g\,d\alpha + \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b f\,d\alpha - \frac{\epsilon}{2} < \int_a^b g\,d\alpha + \frac{\epsilon}{2} \quad [\because S(P,f,\alpha) \le S(P,g,\alpha)]$$

$$\Rightarrow \int_a^b f\,d\alpha < \int_a^b g\,d\alpha + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

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(44)

$\Rightarrow \int_a^b f\,d\alpha < \int_a^b g\,d\alpha + \epsilon$

Since $\epsilon > 0$ is arbitrary, we have

$\int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$

i.e) $\int_a^b f(x)\,d\alpha(x) \leq \int_a^b g(x)\,d\alpha(x)$

Hence proved.

Theorem : 1.15

Assume that α increasing on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then $|f| \in R(\alpha)$ on $[a,b]$ and we have

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

i.e) $\left|\int_a^b f(x)\,d\alpha(x)\right| \leq \int_a^b |f(x)|\,d\alpha(x)$

Proof : Let $f \in R(\alpha)$ on $[a,b]$

To prove : $|f| \in R(\alpha)$ on $[a,b]$ TP: $U(P,|f|,\alpha) - L(P,|f|,\alpha) < \epsilon$

To prove this we use theorem : 1.3 (i) & (ii)

Consider, $U(P,|f|,\alpha) = \sum_{k=1}^{n} (M_k|f| - m_k|f|)\Delta\alpha_k < \epsilon$

$M_k(f) - m_k(f) = \text{l.u.b}\{f(x) : x \in [x_{k-1}, x_k]\}$

$\qquad - \text{g.l.b}\{f(y) : y \in [x_{k-1}, x_k]\}$

$\Rightarrow M_k(f) - m_k(f) = \text{l.u.b}\{f(x) : x \in [x_{k-1}, x_k]\}$

$\qquad - (-\text{l.u.b}\{-f(y) : y \in [x_{k-1}, x_k]\})$

$\qquad = \text{l.u.b}\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$

$\qquad = \text{l.u.b}\{|f(x) - f(y)| : x, y \in [x_{k-1}, x_k]\}$

Since $f(x) - f(y)$ and $f(y) - f(x)$ are the members of $\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$

Now consider,

$M_k(|f|) - m_k(|f|) = \text{l.u.b}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

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$- g.l.b - \{|f(y)| : y \in [x_{k-1}, x_k]\}$

$= l.u.b \{|f(x)| - |f(y)| : x, y \in [x_{k-1}, x_k]\}$ (4)

$= l.u.b \{||f(x)| - |f(y)|| : x, y \in [x_{k-1}, x_k]\}$

$\leq l.u.b \{|f(x) - f(y)| : x, y \in [x_{k-1}, x_k]\}$

$= M_k(f) - m_k(f)$ $\quad \because |x-y| \geq |x| - |y|$

Also $|x-y| \geq ||x| - |y||$

$\Rightarrow M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$

Since α increasing on $[a,b]$, $\Delta\alpha_k > 0$ we have

$$\sum_{k=1}^{n} [M_k(|f|) - m_k(|f|)]\Delta\alpha_k \leq \sum_{k=1}^{n} [M_k(f) - m_k(f)]\Delta\alpha_k$$

$$\Rightarrow \sum_{k=1}^{n} M_k(|f|)\Delta\alpha_k - \sum_{k=1}^{n} m_k(|f|)\Delta\alpha_k \leq \sum_{k=1}^{n} M_k(f)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\Delta\alpha_k$$

$$\Rightarrow U(P,|f|,\alpha) - L(P,|f|,\alpha) \leq U(P,f,\alpha) - L(P,f,\alpha) \longrightarrow (1)$$

Since $f \in R(\alpha)$ on $[a,b]$, 'f' satisfies the Riemann condition.

i.e) $0 \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Using this in (1) we have,

$U(P,|f|,\alpha) - L(P,|f|,\alpha) \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\Rightarrow U(P,|f|,\alpha) - L(P,|f|,\alpha) < \epsilon$

$\Rightarrow$ $|f|$ satisfies the Riemann condition

$\Rightarrow$ $|f| \in R(\alpha)$ on $[a,b]$

Now, to prove that,

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

we know that,

$$-|f(x)| \le f(x) \le |f(x)|$$

(46)

Since $f \in R(\alpha)$ and $|f| \in R(\alpha)$ on $[a,b]$ and

Taking left inequality

$$-|f(x)| \le f(x),$$

By theorem : 14 we have

$$\int_a^b -|f(x)|\, d\alpha(x) \le \int_a^b f(x)\, d\alpha(x)$$

$$\Rightarrow -\int_a^b |f|\, d\alpha \le \int_a^b f\, d\alpha \longrightarrow (2)$$

Since $f \in R(\alpha)$ and $|f| \in R(\alpha)$ on $[a,b]$ and $f(x) \le |f(x)|$,

By theorem : 4 , we have. right inequality,

$$\int_a^b f(x)\, d\alpha(x) \le \int_a^b |f(x)|\, d\alpha(x)$$

$f(x) \le |f(x)|$

$\int_a^b f(x) d\alpha \le \int_a^b |f(x) d\alpha(x)|$

$$\text{ie)} \int_a^b f\, d\alpha \le \int_a^b |f|\, d\alpha \longrightarrow (3)$$

From (2) and (3) we have.

$$-\int_a^b |f|\, d\alpha \le \int_a^b f\, d\alpha \le \int_a^b |f|\, d\alpha.$$

$$\Rightarrow \left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$$

$\because -a < x < a$

$\Rightarrow |x| < a$

$$\left|\int_a^b f(x)\, d\alpha(x)\right| \le \int_a^b |f(x)\, d\alpha(x)|$$

Hence the proof.

theorem:16 .

Assume that α increasing on $[a,b]$ If $f \in R(\alpha)$ on $[a,b]$, then $f^2 \in R(\alpha)$ on $[a,b]$

proof : Given : $f \in R(\alpha)$ on $[a,b]$

To prove : $f^2 \in R(\alpha)$ on $[a,b]$

ie) To prove : f^2 satisfies the Riemann condition.

$$0 \le U(P, f^2, \alpha) - L(P, f^2, \alpha) < \epsilon$$

Now consider,

$$M_k(f^2) = \text{l.u.b}\{f^2(x) : x \in [x_{k-1}, x_k]\}$$
$$= \text{l.u.b}\{(f(x))^2 : x \in [x_{k-1}, x_k]\}$$
$$= \text{l.u.b}\{(|f(x)|)^2 : x \in [x_{k-1}, x_k]\}$$
$$= \{\text{l.u.b}\{|f(x)| : x \in [x_{k-1}, x_k]\}\}^2$$
$$= (M_k |f|)^2 \qquad \because [f(x)]^2 = [|f(x)|]^2$$

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$\therefore M_k(f^2) = [M_k(|f|)]^2$

Similarly, $m_k(f^2) = [m_k(|f|)]^2$

Now consider,

$$M_k(f^2) - m_k(f^2) = (M_k(|f|))^2 - (m_k(|f|))^2$$
$$M_k(f^2) - m_k(f^2) = (M_k(|f|) - m_k(|f|))(M_k(|f|) + m_k(|f|)) \to (1)$$

Since $|f|$ is bounded on $[a, b]$,

$\exists\ M > 0$, such that $|f(x)| \le M,\ \forall x \in [a, b]$

$\Rightarrow M_k(|f|) \le M$ and $m_k(|f|) \le M$.

$\therefore$ (1) becomes

$$M_k(f^2) - m_k(f^2) \le 2M(M_k(|f|) - m_k(|f|))$$

Since α increasing on $[a, b]$, $\Delta\alpha_k > 0$,

$$\therefore \sum_{k=1}^{n} [M_k(f^2) - m_k(f^2)]\Delta\alpha_k \le 2M\sum_{k=1}^{n} (M_k|f| - m_k(|f|))\Delta\alpha_k$$

$$\Rightarrow \sum_{k=1}^{n} M_k(f^2)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f^2)\Delta\alpha_k \le 2M\left\{\sum_{k=1}^{n} M_k(|f|)\Delta\alpha_k - \sum_{k=1}^{n} m_k(|f|)\Delta\alpha_k\right\}$$

$\Rightarrow U(P, f^2, \alpha) - L(P, f^2, \alpha) < 2M \{ U(P, |f|, \alpha) - L(P, |f|, \alpha) \}$ $\rightarrow (2)$

By thm : 15,

$f \in R(\alpha) \Rightarrow |f| \in R(\alpha)$ on $[a, b]$

$\therefore 0 \le U(P, |f|, \alpha) - L(P, |f|, \alpha) < \frac{\epsilon}{2M}$

$\therefore$ (2) becomes

$U(P, f^2, \alpha) - L(P, f^2, \alpha) < 2M \cdot \frac{\epsilon}{2M} < \epsilon$

$\Rightarrow f^2$ satisfies Riemann condition w.r.t α on $[a, b]$

$\Rightarrow f^2 \in R(\alpha)$ on $[a, b]$

Theorem: 17

Assume that α increasing on $[a, b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$, then the product $f \cdot g \in R(\alpha)$ on $[a, b]$.

proof: Given that : $\alpha \uparrow$ on $[a, b]$

~~To prove that~~ : If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b)$ then T.P $f \cdot g \in R(\alpha)$

Consider,

$[f(x) + g(x)]^2 = [f(x)]^2 + [g(x)]^2 + 2f(x) \cdot g(x)$

$\Rightarrow 2f(x)\, g(x) = [f(x) + g(x)]^2 - [f(x)]^2 - [g(x)]^2$

$\Rightarrow f(x) \cdot g(x) = \frac{1}{2} \{ [f(x) + g(x)]^2 - [f(x)]^2 - [g(x)]^2 \}$ $\rightarrow (1)$

Since $f \in R(\alpha)$ on $[a, b]$, $f^2 \in R(\alpha)$ on $[a, b]$

Since $g \in R(\alpha)$ on $[a, b]$; $g^2 \in R(\alpha)$ on $[a, b]$ By thm : 16

Also, $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$

$\Rightarrow f + g \in R(\alpha)$ on $[a, b]$

$\Rightarrow f(x) + g(x) \in R(\alpha)$ on $[a, b]$ (By thm 2 : 1)

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